QUESTIONS AND DISCUSSIONS.

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DISCUSSIONS.

I. RELATING TO GENERALIZATIONS OF THE WITCH AND THE CISSOID.

BY FREDERICK H. HODGE, Franklin College.

Suppose a circle given of radius a tangent to the x-axis at the origin and a second circle of radius $2a + k$ tangent to the first at the point where the y-axis cuts it and, hence, having $(0, -k)$ as the coördinates of its center. We draw a radius vector (see figure), OS, through the origin making an angle ϕ with the x-axis and cutting the circles in the points R and S respectively. We draw a line through R parallel to the x-axis and a line through S parallel to the y-axis. We wish to determine the equation of the locus of P the point of intersection of these two lines as the radius vector revolves about the origin.

From the figure it is readily seen that the parametric equations of the locus are

$$x = \overline{OS} \cos \phi, \qquad y = \overline{OR} \sin \phi.$$

Since $\overline{OR} = 2a \sin \phi$, we have $y = 2a \sin^2 \phi$ and $\sin^2 \phi = y/2a$. To determine the value of $\overline{OS}$ we connect C with S and in the triangle OSC we have, by the law of cosines, $\overline{CS}^2 = (2a + k)^2 = k^2 + \overline{OS}^2 - 2k\,\overline{OS} \cos (90° + \phi) = k^2 + \overline{OS}^2 + 2k\,\overline{OS} \sin \phi$. Hence, $\overline{OS}^2 + 2k\,\overline{OS} \sin \phi - 4a^2 - 4ak = 0$. This gives the desired value for $\overline{OS}$,

$$\overline{OS} = - k \sin \phi \pm \sqrt{4a^2 + 4ak + k^2 \sin^2 \phi},$$

from which we have

$$x = (- k \sin \phi \pm \sqrt{4a^2 + 4ak + k^2 \sin^2 \phi}) \cos \phi.$$

Transposing the $- k \sin \phi \cos \phi$, squaring and simplifying by means of the relation $y = 2a \sin^2 \phi$, we obtain the desired equation:

$$a^2[x^2 - 2(a + k)(2a - y)]^2 = k^2x^2(2ay - y^2).$$

Except for particular values of k this is a fourth-degree equation and the locus passes through the points $(0, 2a)$ and $(\pm 2 \sqrt{a(a + k)}, 0)$.

If $k = 0$ the equation reduces to the parabola $x^2 = - 2ay + 4a^2$.

If $k = \infty$, the larger circle becomes a straight line and the construction is identical with the well-known construction giving the witch of Agnesi. To verify the fact that the witch is a member of the family of curves we have under consideration we take the equation in the form

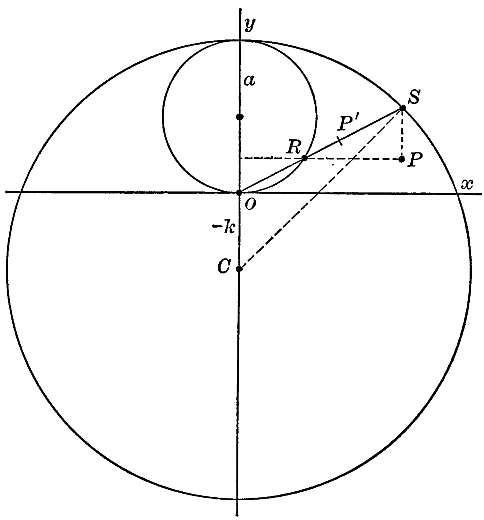
$$x^2 - \frac{kx}{a} \sqrt{2ay - y^2} = 2(a + k)(2a - y),$$

divide by k and then let k approach ∞. The equation then reduces to

$$-x\sqrt{2ay-y^2} = 2a(2a-y) \text{ or } x^2y = 4a^2(2a-y),$$

which is the equation of the witch.

To obtain a generalization of the cissoid, take a radius equal to OR and with S as center cut SO at P' (see figure). We proceed to show that the locus of the point P' as the radius vector revolves about the point O is such a curve.



From the figure,

$$OP' = OS - P'S = OS - OR.$$

From the work above we have

$$OS = -k\sin\varphi \pm \sqrt{4a^2+4ak+k^2\sin^2\varphi},$$

$$OR = 2a\sin\varphi.$$

Hence, the locus of P' is given in polar coördinates by the equation

$$= -(k+2a)\sin\varphi \pm \sqrt{4a^2+4ak+k^2\sin^2\varphi}.$$

Transforming from polar to rectangular coördinates and simplifying, the equation becomes

$$(x^2+y^2)^2 + 2(k+2a)y(x^2+y^2) = 4a(a+k)x^2.$$

This gives a quartic curve very much like the cardioid in appearance. It passes through the points $(0, 0)$, $(2\sqrt{a(a+k)}, 0)$ and $(0, -2\sqrt{k+2a})$. If both sides of the equation be divided by k and k allowed to approach infinity the equation approaches the limiting form $y^3 = x^2(2a-y)$, which is the equation of the cissoid (x and y being interchanged from the usual form). Obviously as k

approaches infinity the larger circle approaches a straight line and the construction merges into the classic construction for the cissoid.

II. Relating to a Geometric Proof of a Theorem on Collineations.

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The relations between double elements of a collineation in space (by which is meant a three-space, unless otherwise stated) have been discussed analytically, use being made of the characteristic equation of the collineation and of the properties of elementary divisors.[1] Enriques, in his *Projective Geometrie* (Leipzig, 1903), has shown geometrically that if a collineation in a plane has a double point it has a double line. It is here proposed to prove the

Theorem. *If a collineation in space has a double point, it has a double plane.*

If the collineation is perspective the theorem is evident. Suppose, then, that the collineation is non-perspective.

Let A be a double point of a non-perspective collineation π and let a be a line through A. Then π sets up the following correspondence

$$a \barwedge a_1 \barwedge a_2 \barwedge a_3 \barwedge a_4.$$

Let B and C be two distinct points on a different from A. Then

$$B \barwedge B_1 \barwedge B_2 \barwedge B_3 \barwedge B_4,$$

where B_i is on a_i $(i = 1, 2, 3, 4)$, and similarly for the C's. Let the intersection $(BB_1, CC_1) = D$. Since A is a self-corresponding point, D will be independent of the points B and C. Also, let $(B_iB_j, C_iC_j) = D_i$, $(i = 1, 2, 3, j = i + 1)$.

If a, a_1 and a_2 are coplanar the theorem is evident and the double plane contains also a double line.

If a, a_1 and a_2 are not coplanar, D_1 is not on plane $[aa_1]$. Let the line DD_1 meet the plane $[aa_1]$ in some point E, and let us assume C so chosen that E lies on CC_1. Let $\pi(E) = E_1$. E_1 is on D_2D_3 and also on the plane $[CC_1, C_1C_2]$. Hence D, D_1, D_2 and D_3 are coplanar and the plane containing them is a double plane.

By duality, *if a collineation in space has a double plane it also has a double point.*

The method here used for establishing the theorem for a three-space is general and can be applied to an n-space giving the theorem that if a collineation in an n-space has a double point it has a double $(n - 1)$-space. This method of proof when applied to collineations in a plane is somewhat simpler than that of Enriques mentioned above.

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[1] Segre, "Sulla teoria e sulla classificazione delle omografie in un spazio lineare ad un numero qualunque di dimensioni," *Reale Accad. dei Lincei*, Serie 3*a*, Bd. XIX, S. 6. See also Muth, *Theorie und Anwendung der Elementartheiler*, Leipzig, 1899; Veblen and Young, *Projective Geometry*, Vol. I, Boston, 1910; Bôcher, *Introduction to Higher Algebra*, N. Y., 1912; Newson, "A New Theory of Collineations," *Amer. Jour. Math.*, Vol. XXIV.